Semantics for coinductive structures in stochastic processes

Infinite product measures in quasi-Borel spaces

Ohad Kammar[1], **Seo Jin Park**[2], Sam Staton[2]

[1]University of Edinburgh, [2]University of Oxford

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Overview

1. **Random streams - a coalgebraic interpretation**
2. Quasi-Borel spaces
3. Infinite product measures in QBS

Compositional Probability [Giry, 1982]

category of spaces

$$\begin{tikzcd} \mathbf{Space} \arrow[loop below, "P"] \end{tikzcd}$$

strong monad

Monadic probabilistic programs define probabilistic models.
Space and P may be:

- Polish spaces and Borel/Radon measures
- Standard Borel spaces and probability measures

This talk:

- quasi-Borel spaces (QBS) and probability distributions [Heunen et al., 2017]

Stochastic Processes

$$
\begin{tikzcd}
 & \text{Discrete-time stochastic processes} \arrow[dl, "\text{e.g.}"'] \arrow[dr, "\text{e.g.}"] & \\
\text{i.i.d sequences} \arrow[dr] & & \text{Markov chains} \arrow[dl] \\
 & \text{Distributions in } P(B^{\mathbb{N}}) &
\end{tikzcd}
$$

Many stochastic processes are generated step by step.

$p \in PB$ $\qquad\qquad$ $b \in PB,\ f : B \to PB$

$p \quad p \quad p \quad \cdots$ $\qquad\qquad$ $p \xrightarrow{f} p \xrightarrow{f} p \to \cdots$

Stochastic Processes as Random Streams

Deterministic streams	$B^{\mathbb{N}}$	Universal property [Rutten, 2000]: final coalgebra of $X \mapsto B \times X$ $\mathsf{step} : B^{\mathbb{N}} \xrightarrow{\langle \mathsf{head}, \mathsf{tail} \rangle} B \times B^{\mathbb{N}}$
Random streams	$P(B^{\mathbb{N}})$	$B^{\mathbb{N}}$ is final coalgebra in Kleisli(P) in QBS for standard-Borel spaces, but it is open in general

$B^{\mathbb{N}}$ is the final coalgebra of $X \mapsto B \times X$ in QBS:

$$\mathsf{step} : B^{\mathbb{N}} \xrightarrow{\langle \mathsf{head}, \mathsf{tail} \rangle} B \times B^{\mathbb{N}}$$

$B^{\mathbb{N}}$ is the final coalgebra of $X \mapsto B \times X$ in QBS:

$$\text{step} : B^{\mathbb{N}} \xrightarrow{\langle \text{head}, \text{tail} \rangle} B \times B^{\mathbb{N}}$$

Question: Is $B^{\mathbb{N}}$ also the final coalgebra of $X \mapsto B \times X$ in Kleisli(P)?
For all Kleisli morphisms

$$S \xrightarrow{f} P(B \times S)$$

does there exist a unique Kleisli morphism

$$\text{chain}(f) : S \to P(B^{\mathbb{N}})$$

such that

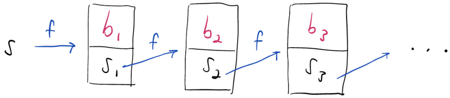
$$\begin{tikzcd}
S \arrow[r, "f"] \arrow[d, dashed, "\text{chain}(f)"'] & B \times S \arrow[d, dashed, "id \times \text{chain}(f)"] \\
B^{\mathbb{N}} \arrow[r, "\text{step}"] & B \times B^{\mathbb{N}}
\end{tikzcd}$$

As a probabilistic program (LazyPPL - Haskell):

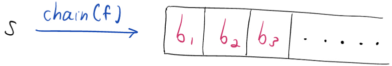
```
chain :: (S -> Prob (B,S))
       -> S -> Prob (Stream B)
chain f s = do {(b,s') <- f s;
                bs <- chain f s';
                return (b:bs)}
```

Iteratively applies f in an infinite chain

$$f : S \to P(B \times S)$$



$$\mathrm{chain}(f) : S \to P(B^{\mathbb{N}})$$



Question: Is chain(f) well-defined for any S and f?

Infinite Product Measures in QBS

This talk: initial progress in proving that for all qbs B, $\text{chain}(f) : S \to P(B^{\mathbb{N}})$ is a well-defined qbs morphism for

$$S = (PB)^{\mathbb{N}}$$

$$f : (PB)^{\mathbb{N}} \to P(B \times (PB)^{\mathbb{N}})$$

$$(p_n)_{n\geq 1} \mapsto p_1 \otimes \delta(p_2, p_3, \ldots)$$

Then $\text{chain}(f)$ describes the stateless processes.

$$(p_n)_{n \geq 1} \subseteq PB$$

$$p_1 \quad p_2 \quad p_3 \quad \cdots \quad p_n \quad \cdots$$

$$\begin{tikzcd}
(PB)^{\mathbb{N}} \arrow[rr, "(p_n)_{n\geq 1}\mapsto p_1\otimes\delta_{(p_2,p_3,\dots)}"] \arrow[d, dashed, "\mathrm{chain}(f)"] & & B\times(PB)^{\mathbb{N}} \arrow[d, dashed, "id\times\mathrm{chain}(f)"] \\
B^{\mathbb{N}} \arrow[rr, "\mathrm{step}"] & & B\times B^{\mathbb{N}}
\end{tikzcd}$$

$$\mathrm{chain}(f)(p_1, p_2, p_3, \dots) = p_1 \otimes \mathrm{chain}(f)(p_2, p_3, \dots)$$

Program interpretation:

```
prod :: Stream (Prob b) -> Prob (Stream b)
prod (p:ps) = do {x <- p; xs <- prod ps; return (x:xs)}
```

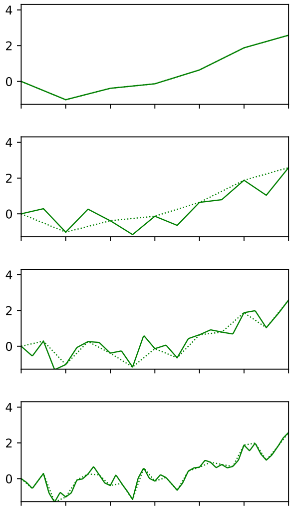
Intuition: the sequence of measures $(p_n)_{n\geq 1}$ are *independently stitched together*.

Application: Brownian motion in QBS

- [Karatzas and Shreve, 1998]: approximate Brownian motion by incrementally adding up a sequence of independent random functions.
- In QBS, can coinductively generate a random stream of functions

$$P((\mathbb{R}^{[0,T]})^{\mathbb{N}})$$

approximating Brownian motion.



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Higher order functions in probability

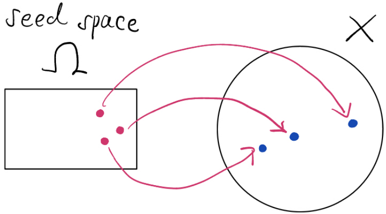
Theorem ([Aumann, 1961])

The category of measurable spaces is not Cartesian-closed.

Hence measure spaces cannot give semantics to higher-order PPLs.

Solution: **Quasi-Borel spaces (QBS)**

Random elements in X are represented by $\Omega \to X$ where $\Omega = [0, 1]$.



- Measurable spaces use σ-algebras to implicitly specify which $\Omega \to X$ to admit as random elements.
- Quasi-Borel spaces axiomatise the admissible random elements $\Omega \to X$.

Quasi-Borel Spaces [Heunen, K, S, Yang, 2017]

Definition

A **quasi-Borel space** (X, M_X) is:

- X: set of points
- $M_X \subset [\Omega \to X]$: set of **admissible random elements**, such that:
  - it contains all the constant functions,
  - $\alpha \in M_X$, $f : \Omega \to \Omega$ measurable $\implies \alpha \circ f \in M_X$,
  - $\Omega = \bigcup_{i \in \mathbb{N}} S_i$ is Borel partition, $\alpha_i \in M_X, i \in \mathbb{N} \implies \lambda r.\text{case } r \text{ of } r \in S_i \to \alpha_i(r) \in M_X$.

where $\Omega = [0, 1]$.

Quasi-Borel Spaces [Heunen, K, S, Yang, 2017]

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where $\Omega = [0, 1]$.

Definition

A map $f : X \to Y$ is a **qbs morphism** if

$$(\Omega \xrightarrow{\alpha} X) \in M_X \implies (\Omega \xrightarrow{\alpha} X \xrightarrow{f} Y) \in M_Y$$

Quasi-Borel Spaces [Heunen, K, S, Yang, 2017]

Quasi-Borel spaces form a category **QBS**.
Properties of QBS:

- QBS is **Cartesian-closed**.
- QBS contains **standard-Borel spaces** as a full subcategory.
- QBS supports a **strong commutative probability monad** suitable for probabilistic programming.

QBS Probability Monad

$$\Sigma_{M_X} := \{U \subset X \mid \forall \alpha \in M_X.\ \alpha^{-1}U \in \Sigma_\Omega\}$$

Fix a probability measure μ on (Ω, Σ_Ω). Then $\forall \alpha \in M_X$,

$$\boxed{\begin{aligned} \boldsymbol{\alpha_*\mu} : \Sigma_{M_X} &\to [0,1] \\ U &\mapsto \mu(\alpha^{-1}U) \end{aligned}}$$

is a probability measure on Σ_{M_X}, called the **pushforward measure** of α w.r.t μ.

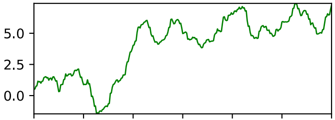
$$\int_X k \ d\alpha_*\mu = \int_\Omega (k \circ \alpha) \ d\mu$$

$\alpha, \beta \in M_X$ are **equal in law** if $\alpha_*\mu = \beta_*\mu$ as measures on Σ_{M_X}.

$$\boxed{PX := M_X/\sim \ \text{ where } \ \sim\ = \ \text{ law equality}}$$

Random Functions

- **Random functions** are distributions on functions.
  - e.g. Brownian motion, stochastic differential equations (random functions $[0, \infty) \to \mathbb{R}$).



  - Defined as parametrised family of random variables $(B_t)_{t \geq 0}$, $B_t : \Omega \to \mathbb{R}\ \forall t \geq 0$.
- Random functions $X \to Y$ are explicitly modelled by $P(X^Y)$ in QBS.
- Cartesian-closure of QBS allows us to venture beyond standard probability to directly use function spaces.

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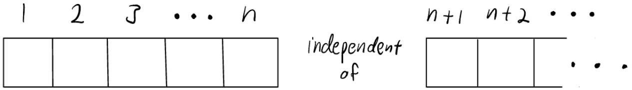
Defining $\mathsf{Ind}B^{\mathbb{N}}$

Characterising independence in QBS:

1. every finite prefix is independent of its suffix



2. satisfies a 0-1 law

Defining $\mathrm{Ind} B^{\mathbb{N}}$

The strength and commutativity of P gives the operation

$$\otimes : PX \times PY \to P(X \times Y)$$

Hence for all finite n we have an operation

$$\begin{aligned} (PB)^n &\to P(B^n) \\ (p_1, ..., p_n) &\mapsto p_1 \otimes \cdots \otimes p_n \end{aligned}$$

Independence for infinite n?

QBS Products

The sigma algebra for the QBS product cannot be characterised by the product sigma algebra.

$$\bigotimes_n \Sigma_{M_{X_n}} \subsetneq \Sigma_{M_{\prod_n X_n}}$$

Defining $\mathrm{Ind}B^{\mathbb{N}}$

Definition (0-1 law)

For all $n \geq 1$, define the sub sigma algebras of $\Sigma_{B^{\mathbb{N}}}$:

$$\Sigma_n := \{\prod_{i=1}^{n} X_i \times S \mid S \in \Sigma_{\prod_{i \geq n+1} X_i}\}$$

$\bigcap_{n \geq 1} \Sigma_n$ is called the **tail sigma algebra**. Say E is a **tail event** if $E \in \bigcap_{n \geq 1} \Sigma_n$.
$p \in P(B^{\mathbb{N}})$ satisfies the **0-1 law** if

$$\forall E \in \bigcap_{n \geq 1} \Sigma_n,\ p(E) \in \{0, 1\}$$

$\bigcap_{n \geq 1} \Sigma_n$ is the sigma algebra of events invariant under finitely many changes.

Defining $\mathrm{Ind}B^{\mathbb{N}}$

Definition

Define $\mathrm{Ind}B^{\mathbb{N}} \subset P(B^{\mathbb{N}})$ as the subspace consisting of those measures $p \in P(B^{\mathbb{N}})$ such that:

1. There exist $\alpha_i \in M_{X_i}, i \geq 1$ such that for all n,

$$p = \alpha_{1*}\mu \otimes \ldots \otimes \alpha_{n*}\mu \otimes (p \triangleright \pi_{n+1,n+2,\ldots})$$

2. p satisfies the 0-1 law.

Constructing prod : $(PB)^{\mathbb{N}} \to \operatorname{Ind} B^{\mathbb{N}}$

Package a sequence of admissible random elements $(\alpha_n : \Omega \to B)_{n \geq 1}$ into a single admissible random element for $\operatorname{Ind} B^{\mathbb{N}} \subset P(B^{\mathbb{N}})$:

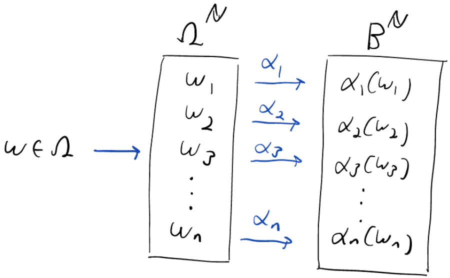
$$\boxed{\Omega \cong \Omega^{\mathbb{N}} \xrightarrow{\alpha_1 \times \alpha_2 \times \cdots} B^{\mathbb{N}}}$$

$$(\Omega \cong \Omega^{\mathbb{N}} \quad \because \Omega = [0,1] \text{ is standard-Borel})$$

$$\text{ind} : (B^{\Omega})^{\mathbb{N}} \to (B^{\mathbb{N}})^{\Omega}$$

$$(\alpha_n)_{n \geq 1} \mapsto (\Omega \cong \Omega^{\mathbb{N}} \xrightarrow{\alpha_1 \times \alpha_2 \times \cdots} B^{\mathbb{N}})$$

depends only on law $\alpha_{i*}\mu$.



Theorem (Infinite product measures in QBS)

There exists a unique morphism $\mathrm{prod} : (PB)^{\mathbb{N}} \to \mathrm{Ind}B^{\mathbb{N}}$ *such that the diagram below commutes:*

$$\begin{tikzcd}
(B^{\Omega})^{\mathbb{N}} \arrow[r, two heads, "\mathrm{law}^{\mathbb{N}}"] \arrow[d, "\mathrm{law}\circ\mathrm{ind}"'] & (PB)^{\mathbb{N}} \arrow[dl, dashed, "\mathrm{prod}"'] \arrow[d, "{(p_n)_n \mapsto (p_1\otimes\cdots\otimes p_n)_n}"] \\
\mathrm{Ind}B^{\mathbb{N}} \arrow[r, tail, "{p\mapsto(p\triangleright\pi_{1,\ldots,n})_n}"'] & \prod_{n\geq 1} P(B^n)
\end{tikzcd}$$

where

$$\mathrm{law} : B^{\Omega} \to PB$$
$$\alpha \mapsto \alpha_*\mu$$

Theorem (0-1 law for Ind)

For all $\vec{p} \in (PB)^{\mathbb{N}}$, $\mathrm{prod}(\vec{p})(E) \in \{0,1\}$ *for all tail events* E.

I.I.D Sequences in QBS

We also get an analogue of **i.i.d** sequences in QBS as a corollary.

$$\begin{aligned} \mathrm{iid} : PB &\to \mathrm{Ind}\, B^{\mathbb{N}} \\ p &\mapsto \mathrm{prod}(p, p, \ldots) \end{aligned}$$

$PB \xrightarrow{\mathrm{iid}} \mathrm{Ind}\, B^{\mathbb{N}} \hookrightarrow P(B^{\mathbb{N}})$ gives semantics to the program

```
iid :: Prob b -> Prob (Stream b)
iid p = do {x <- p; xs <- iid p; return (x:xs)}
```

Theorem (Hewitt-Savage 0-1 law)

For all $p \in PX$, $\mathrm{iid}(p)$ satisfies the 0-1 law.

Future Work

- Full final coalgebra theorem for streams in Kleisli(P) in QBS?
- Analogue of Ionescu-Tulcea theorem [Kallenberg, 2002] for Kleisli(P) in QBS?

Summary

- Coalgebraic picture of discrete-time stochastic processes in QBS: $B^{\mathbb{N}}$ as the final $(B \times -)$-coalgebra in Kleisli(P).

$$\mathrm{chain}(f) : S \to P(B^{\mathbb{N}})$$

- Infinite product measures in QBS:

$$\mathrm{Ind}B^{\mathbb{N}} \subset P(B^{\mathbb{N}}),\ \mathrm{prod} : (PB)^{\mathbb{N}} \to \mathrm{Ind}B^{\mathbb{N}}$$

```
prod :: Stream (Prob b) -> Prob (Stream b)
prod (p:ps) = do {x <- p; xs <- prod ps; return (x:xs)}
```

- Applications: random stream of functions $P((\mathbb{R}^{[0,T]})^{\mathbb{N}})$ approximating Brownian motion in QBS.

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