Semantics for the coinductive structures in stochastic processes: Infinite product measures in quasi-Borel spaces

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1 Introduction

A compositional view of probabilistic modelling is that probability can be modelled by a monad [4] and probabilistic models are built by composing Kleisli morphisms. In this talk we will explore the relevance of that view to stochastic processes, that include both random streams and random functions.

Random streams via final coalgebras: Deterministic streams are modelled by infinite sequences X^ω, which have the universal property of final coalgebras [11]. *Random* streams come up naturally as discrete-time stochastic processes, such as Markov chains and i.i.d sequences. Kerstan and König framed this in terms of final coalgebras in the Kleisli category of the Giry monad [10]. A common example is the infinite product measure.

Random functions via quasi-Borel spaces: Deterministic function spaces are usually modelled by Cartesian-closed categories. *Random* functions are distributions on function spaces, such as Brownian motion. The category of quasi-Borel spaces (QBS) is Cartesian-closed while supporting a probability monad P and the real numbers $\mathbb{R}$ [5].

In this talk, we will address an open question (since 2017) of whether the Kleisli category for the probability monad P on quasi-Borel spaces supports random streams in general. We have initial progress in that we can give a semantics to the programs `prod` and `iid` in Figure 1.

Figure 1 Programs for product measures and iid sequences

```
prod :: Stream (Prob B) -> Prob (Stream B)
prod (p:ps) = do {x <- p; xs <- prod ps; return (x:xs)}

iid :: Prob A -> Prob (Stream A)
iid p = do {x <- p; xs <- iid p; return (x:xs)}
```

Combining these two directions allows us to consider random streams of functions. One source of examples are discrete-time stochastic processes that converge to a random function. In figure 2, Lévy's construction of Brownian motion [9] is used to generate a random stream of functions in $P((\mathbb{R}^{\mathbb{R}})^\omega)$. The random stream converges to Brownian motion. This gives a way to approximate Brownian motion in LazyPPL, a probabilistic programming language that supports random streams and random functions [2].

2 On quasi-Borel spaces

A quasi-Borel space is a set X together with a set $M_X \subseteq [\Omega \to X]$ of admitted random elements. Here Ω is an uncountable standard probability space and can be taken as $[0, 1]$.

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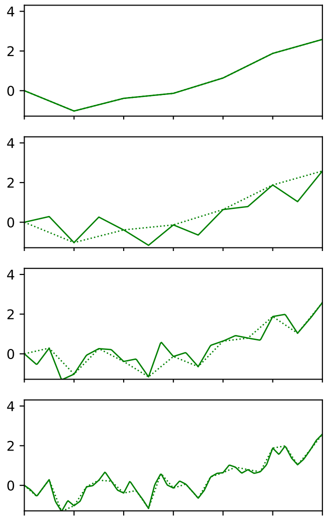
Quasi-Borel spaces form a Cartesian closed category (and in fact a quasitopos). This is their big advantage over ordinary measurable spaces or Borel spaces. Moreover, they contain standard Borel spaces (such as $\mathbb{R}$, $\mathbb{N}$) as a full subcategory. However, to achieve Cartesian closure, the categorical products in general quasi-Borel spaces cannot be characterized using product σ-algebras, and so some of the machinery of stochastic processes has to be revisited.

Even though we cannot talk about product σ-algebras for arbitrary QBS, the probability monad P on QBS is commutative, and we can define *finite* product measures using this commutative structure, e.g. $\otimes : P(A) \times P(B) \to P(A \times B)$.

In more detail, the probability monad is defined as follows. If $k : X \to \mathbb{R}$ is a quasi-Borel morphism and $\alpha : \Omega \to X$ is an admitted random element then $(k \circ \alpha) : \Omega \to \mathbb{R}$ is necessarily an ordinary Borel function, and so we can calculate $\int_\Omega (k \circ \alpha) \in \mathbb{R}$. We say $\alpha \sim \beta$ if for all $k : X \to \mathbb{R}$, $\int_\Omega (k \circ \alpha) = \int_\Omega (k \circ \beta)$. The probability monad $P(X)$ comprises admitted random elements, modulo $\sim$.

Figure 2 One sample $(f_n)_n \sim$ BM from the random stream of functions approximating Brownian motion. Only the first four items in the stream are plotted. At each step, the previous step is overlaid with a dotted line.



3 Towards a universal property

For any QBS B, there is a standard construction of the QBS B^ω of infinite sequences: it is the final coalgebra of the endofunctor $X \mapsto B \times X$. It is still an open question whether B^ω is a final coalgebra in Kleisli(P). Our main result says that there *is* the dotted coalgebra homomorphism in the following diagram in Kleisli(P).

$$\begin{tikzcd}
(PB)^\omega \arrow[r, "(m_n)_n \mapsto m_1 \otimes \delta_{(m_2, m_3, \ldots)}"] \arrow[d, dashed, "\text{prod}"] & B \times (PB)^\omega \arrow[d, dashed, "(b\, \vec{m}) \mapsto \delta_b \otimes f(\vec{m})"] \\
B^\omega \arrow[r, "\vec{b} \mapsto \delta_{\vec{b}}"] & B \times B^\omega
\end{tikzcd}$$

The Kleisli morphism that is the map to the final coalgebra in the diagram, $f : (PB)^\omega \to P(B^\omega)$, satisfies $f(m_1, m_2, \ldots) = m_1 \otimes f(m_2, m_3, \ldots)$, which means f is exactly the morphism giving semantics to `prod` in Fig. 1, the infinite product measure.

To construct (prod), we consider an infinite sequence of admitted random elements $\alpha_n : \Omega \to B$, representing a sequence of $\sim$-equivalence classes in $(PB)^\omega$, and package it up as an admitted random element

$$\Omega \;\cong\; \Omega^\omega \xrightarrow{(\alpha_n)_n} B^\omega.$$

representing an $\sim$-equivalence class in $P(B^\omega)$. The challenge is to show that this transformation respects equivalence classes in $P(B)$.

To show this, and prove that prod : $(PB)^\omega \to P(B^\omega)$ is a valid morphism, we revisit concepts of tail σ-algebra and conditional expectation to establish a new 0-1 law for quasi-Borel spaces. See the appendix for details.

4 Future Work

It remains open to prove a full final coalgebra theorem for streams in QBS. This would require an analogue of the Ionescu-Tulcea theorem [7] for quasi-Borel spaces.

Infinite products appear in Markov categories [3] and in stochastic memoization [6]. The `prod` program may form a weak distributive law (e.g. [1, §4.5.3]).

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A Overview of independent product measures in QBS, and the 0-1 law

Here we formally define the class of infinite-dimensional measures with independently distributed components in QBS, and give them a well-defined construction.

Fix a probability measure μ on Ω. Let $(X_n, M_{X_n}), n \geq 1$ be a sequence of QBSes. Let $\Sigma_n := \{\prod_{i=1}^{n} X_i \times S \mid S \in \Sigma_{\prod_{i \geq n+1} X_i}\}$ for all n. $\bigcap_n \Sigma_n$ is called the **tail sigma algebra**. We say that $m \in P(\prod_n X_n)$ satisfies the **0-1 law** if every $E \in \bigcap_n \Sigma_n$, $m(E) \in \{0, 1\}$.

Define $\mathrm{Ind}\vec{X} \subset P(\prod_n X_n)$ as the subspace consisting of those measures $m \in P(\prod_n X_n)$ such that:

1. There exist $\zeta_i \in M_{X_i}, i \geq 1$ such that for all n,
$$m = \zeta_{1*}\mu \otimes \ldots \otimes \zeta_{n*}\mu \otimes (m \triangleright \pi_{n+1,n+2,\ldots})$$
2. m satisfies the 0-1 law.

Then we obtain the following results (detailed proofs are in the appendix).

▶ **Theorem 1** (Characterisation of Ind). *There exists a unique morphism prod* : $\prod_n P(X_n) \to$ *$Ind\vec{X}$ such that the diagram below commutes:*

$$\begin{tikzcd}
\prod_{n\geq 1} R(X_n) \arrow[r, two heads, "(\zeta_n)_n \mapsto (\zeta_n * \mu)_n"] \arrow[d, "(\zeta_n)_n \mapsto"', "(\zeta_1 \times \zeta_2 \times \cdots)_* \mu^{\mathbb{N}}"'] & \prod_{n\geq 1} P(X_n) \arrow[d, "(m_n)_n \mapsto", "(m_1 \otimes \cdots \otimes m_n)_n"] \arrow[dl, dashed, "prod"'] \\
Ind\vec{X} \arrow[r, tail, "m \mapsto (m \triangleright \pi_{1,\ldots,n})_n"'] & \prod_{n\geq 1} P(\prod_{i=1}^{n} X_i)
\end{tikzcd}$$

▶ **Theorem 2** (0-1 law for Ind). *For all $\vec{m} \in \prod_{n\geq 1} P(X_n)$, $prod(\vec{m})$ satisfies the 0-1 law.*

B Construction

Here we prove that the construction of $m \in \mathrm{Ind}\vec{X}$ is well-defined.

Let RX be the QBS X^{Ω}. It is known that the pushforward morphism $RX \to PX$ given by $\alpha \mapsto \alpha_* \mu$ is a strong epimorphism.

▶ **Lemma 3.** *For $A \in \Sigma_{\Omega^{\mathbb{N}}}$, if for all n there exists $B_n \in \Sigma_{\prod_{i\geq n+1}\Omega}$ such that $A = \Omega^n \times B_n$, then $\mu^{\mathbb{N}}(A) \in \{0, 1\}$.*

▶ **Lemma 4.** *$(\zeta_1 \times \zeta_2 \times \cdots)_* \mu^{\mathbb{N}} \in Ind\vec{X}$ for all $\zeta_i \in M_{X_i}$.*

▶ **Lemma 5.** *If $m \in Ind\vec{X}$, $\mathbb{E}_m[k \mid \Sigma_n](\vec{x}) = \int k(y_1, ..., y_n, x_{n+1}, x_{n+2}, ...)\ m_n(d(y_1, ..., y_n))$ for all measurable $k : \vec{X} \to [0, \infty)$, where $m_n := m \triangleright \pi_{1,\ldots,n}$.*

Then this implies that if $m, m' \in \mathrm{Ind}\vec{X}$ and their finite-dimensional marginals are equal, then $\mathbb{E}_m[k \mid \Sigma_n] = \mathbb{E}_{m'}[k \mid \Sigma_n]$ a.s. for all n. By Levy's downward martingale convergence theorem [8], $\mathbb{E}_m[k \mid \Sigma_n] \to \mathbb{E}_m[k \mid \bigcap_n \Sigma_n]$ a.s. as $n \to \infty$. This holds for m' as well, hence $\mathbb{E}_m[k \mid \bigcap_n \Sigma_n] = \mathbb{E}_{m'}[k \mid \bigcap_n \Sigma_n]$ a.s.. As m has the 0-1 law, $\mathbb{E}_m[k \mid \bigcap_n \Sigma_n]$ is constant a.s. hence $\mathbb{E}_m[k \mid \bigcap_n \Sigma_n] = \int k\ dm$ a.s.. Similarly holds for m', hence $\int k\ dm = \int k\ dm'$. As k was arbitrary, this means $m = m'$.

This leads to the results below.

▶ **Lemma 6.** *The map $Ind\vec{X} \rightarrowtail \prod_{n\geq 1} P(\prod_{i=1}^{n} X_i)$ defined $m \mapsto (m \triangleright \pi_{1,\ldots,n})_{n\in\mathbb{N}}$ is a monomorphism.*

Hence this proves Theorem 1. Theorem 2 follows immediately.

This gives semantics to **prod** in Figure 1.

C i.i.d measures and Hewitt-Savage 0-1 Law

Let (X, M_X) be a QBS. Define $\mathrm{IID}X \subset P(X^{\mathbb{N}})$ to be the subspace consisting of those measures m such that for all n, there exists $a \in PX$ such that

$$m = \mathrm{iid}_n a \otimes m$$

where $\mathrm{iid}_n : PX \to PX^n$ are defined inductively by:

$$\mathrm{iid}_1 p := p$$
$$\mathrm{iid}_{n+1} p = \mathrm{iid}_n p \otimes p$$

As a consequence of Theorem 1 and Theorem 2, we have the below results.

▶ **Theorem 7** (Characterisation of IID). *There exists a unique QBS morphism $iid : PX \to IIDX$ such that the diagram below commutes:*

$$\begin{tikzcd}
& PX \arrow[dl, dashed, "iid"'] \arrow[d, "a \mapsto (iid_n a)_n"] \\
IIDX \arrow[r, tail, "m \mapsto (m \triangleright \pi_{1,\dots,n})_n"'] & \prod_n P(X^n)
\end{tikzcd}$$

▶ **Theorem 8** (Hewitt-Savage 0-1 law). *For all $m \in PX$, $iid(m)$ satisfies the 0-1 law.*